# UPSC IFS
# (Mains)

## Previous Year Paper
### (Statistics - I)
### 01 Mar, 2022

Indian Forest Service (Main) Exam, 2021

ZCVB-U-STSC

# STATISTICS

## Paper – I

Time Allowed : **Three Hours**

Maximum Marks : **200**

---

### Question Paper Specific Instructions

***Please read each of the following instructions carefully before attempting questions :***

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

*Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.*

*All questions carry equal marks. The number of marks carried by a question/part is indicated against it.*

*Unless otherwise mentioned, symbols and notations have their usual standard meanings.*

*Assume suitable data, if necessary and indicate the same clearly.*

*Answers must be written in **ENGLISH** only.*

---

## SECTION A

**Q1.** (a) (i) Let $P(A_n) = 0$ for each $n \geq 1$. With proper justification show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 0.$$

(ii) Let $P(A_n) = 1$ for each $n \geq 1$. With proper justification show that

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1.$$

*6+2=8*

(b) Do the following functions define cumulative distribution functions ?

(i) $$F(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ x & \text{if} \quad 0 \leq x < \frac{1}{2}, \\ 1 & \text{if} \quad x \geq \frac{1}{2}. \end{cases}$$

(ii) $$F(x) = \begin{cases} 0 & \text{if} \quad x \leq 1, \\ 1 - \frac{1}{x} & \text{if} \quad x > 1. \end{cases}$$

*4+4=8*

(c) Construct sequential probability ratio test for testing $H_0 : p = p_0$ against $H_1 : p = p_1$, when X is a Bernoulli random variable defined with probability function given by

$$f(x; p) = p^x (1 - p)^{1-x}, \; x : 0, 1; \; 0 < p < 1.$$

*8*

(d) A company wishes to purchase one of five different machines : A, B, C, D or E. In an experiment designed to determine whether there is a performance difference among the machines, five experienced operators were assigned to work on the machines for equal times. Data given below represents the number of units produced by each machine :

| | | | | | |
|---|---|---|---|---|---|
| A | 68 | 72 | 77 | 42 | 53 |
| B | 72 | 53 | 63 | 53 | 48 |
| C | 60 | 82 | 64 | 75 | 72 |
| D | 48 | 61 | 57 | 64 | 50 |
| E | 64 | 65 | 70 | 68 | 53 |

Test the hypothesis that there is no difference between the machines at 0·05 significance level.

*8*

[Note : Refer the chi-square table for the theoretical value of $\chi^2$].

Chi-square distribution table is provided at the end of the booklet.

(e) Let $X_1$ and $X_2$ be independent and identically distributed random variables with common probability mass function

$$P[X = \pm 1] = \frac{1}{2}.$$

Write $X_3 = X_1 X_2$. Show that $X_1, X_2, X_3$ are pairwise independent but not independent. *8*

**Q2.** (a) (i) A lot of five identical batteries is life tested. The probability assignment is assumed to be

$$P(A) = \int_A \frac{1}{\lambda} e^{-x/\lambda} dx$$

for any event $A \subseteq [0, \infty)$, where $\lambda > 0$ is a known constant. Thus the probability that a battery fails after time t is given by

$$P(t, \infty) = \int_t^\infty \frac{1}{\lambda} e^{-x/\lambda} dx, \quad t \geq 0.$$

If the times to failure of the batteries are independent, what is the probability that at least two batteries will be operating after $t_0$ hours ? *7*

(ii) Let $(X_1, X_2, ..., X_n)$ be a random sample of size n drawn from a population whose probability density function is given by the form

$$f(x; \theta) = (1 + \theta) x^\theta, \quad 0 < x < 1; \quad \theta > 0.$$

Find the maximum likelihood estimator for $\theta$ and examine whether such an estimator is sufficient for $\theta$. *5+3=8*

(b) Consider a Markov chain assuming values in the state space $S = \{0, 1, 2, 3, 4\}$ and having the following transition probability matrix :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(i) Determine the period of each of the states.

(ii) Identify the states that are transient and the ones that are recurrent. *10*

(c) (i) A practitioner is interested in testing whether the number of accidents occurring daily in a certain industrial plant is Poisson distributed with some unknown mean λ. Assume that the outcome of the number of accidents on a given day is in Region 1 if there are 0 accidents, Region 2 if there is 1 accident, Region 3 if there are 2 or 3 accidents, Region 4 if there are 4 or 5 accidents, and Region 5, if there are more than 5 accidents. Under these given conditions, if the weekly number of accidents over a 30-week period is as given below, test the hypothesis that the number of accidents in a week has a Poisson distribution. *8*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 0 | 0 | 1 | 3 | 4 | 0 | 2 | 12 | 5 |
| 1 | 8 | 0 | 2 | 0 | 1 | 9 | 3 | 4 | 5 |
| 3 | 3 | 4 | 7 | 4 | 0 | 1 | 2 | 1 | 2 |

[Note : Refer the chi-square table for the theoretical value of $\chi^2$]

Chi-square distribution table is provided at the end of the booklet.

(ii) In an experiment of testing the nature of 30 items drawn from a lot, the following sequence of defectives (D) and non-defectives (N) is obtained :

N D D N D N N N D N N D D

N D N D N N D N D D N D

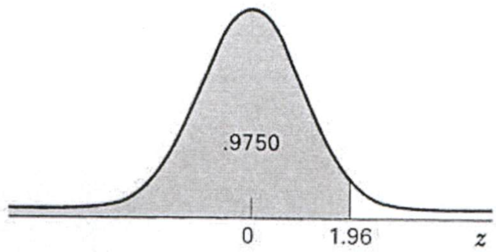
N N D N D

(1) Determine the number of runs.

(2) Examine whether the sequence is random at 5% level of significance. *2+5=7*

[Note : Refer the standard normal table for the theoretical value]

# Normal Distribution Table




| z | −0.09 | −0.08 | −0.07 | −0.06 | −0.05 | −0.04 | −0.03 | −0.02 | −0.01 | 0.00 | z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| −3.80 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | −3.80 |
| −3.70 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | −3.70 |
| −3.60 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0002 | .0002 | −3.60 |
| −3.50 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | −3.50 |
| −3.40 | .0002 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | −3.40 |
| −3.30 | .0003 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0005 | .0005 | .0005 | −3.30 |
| −3.20 | .0005 | .0005 | .0005 | .0006 | .0006 | .0006 | .0006 | .0006 | .0007 | .0007 | −3.20 |
| −3.10 | .0007 | .0007 | .0008 | .0008 | .0008 | .0008 | .0009 | .0009 | .0009 | .0010 | −3.10 |
| −3.00 | .0010 | .0010 | .0011 | .0011 | .0011 | .0012 | .0012 | .0013 | .0013 | .0013 | −3.00 |
| −2.90 | .0014 | .0014 | .0015 | .0015 | .0016 | .0016 | .0017 | .0018 | .0018 | .0019 | −2.90 |
| −2.80 | .0019 | .0020 | .0021 | .0021 | .0022 | .0023 | .0023 | .0024 | .0025 | .0026 | −2.80 |
| −2.70 | .0026 | .0027 | .0028 | .0029 | .0030 | .0031 | .0032 | .0033 | .0034 | .0035 | −2.70 |
| −2.60 | .0036 | .0037 | .0038 | .0039 | .0040 | .0041 | .0043 | .0044 | .0045 | .0047 | −2.60 |
| −2.50 | .0048 | .0049 | .0051 | .0052 | .0054 | .0055 | .0057 | .0059 | .0060 | .0062 | −2.50 |
| −2.40 | .0064 | .0066 | .0068 | .0069 | .0071 | .0073 | .0075 | .0078 | .0080 | .0082 | −2.40 |
| −2.30 | .0084 | .0087 | .0089 | .0091 | .0094 | .0096 | .0099 | .0102 | .0104 | .0107 | −2.30 |
| −2.20 | .0110 | .0113 | .0116 | .0119 | .0122 | .0125 | .0129 | .0132 | .0136 | .0139 | −2.20 |
| −2.10 | .0143 | .0146 | .0150 | .0154 | .0158 | .0162 | .0166 | .0170 | .0174 | .0179 | −2.10 |
| −2.00 | .0183 | .0188 | .0192 | .0197 | .0202 | .0207 | .0212 | .0217 | .0222 | .0228 | −2.00 |
| −1.90 | .0233 | .0239 | .0244 | .0250 | .0256 | .0262 | .0268 | .0274 | .0281 | .0287 | −1.90 |
| −1.80 | .0294 | .0301 | .0307 | .0314 | .0322 | .0329 | .0336 | .0344 | .0351 | .0359 | −1.80 |
| −1.70 | .0367 | .0375 | .0384 | .0392 | .0401 | .0409 | .0418 | .0427 | .0436 | .0446 | −1.70 |
| −1.60 | .0455 | .0465 | .0475 | .0485 | .0495 | .0505 | .0516 | .0526 | .0537 | .0548 | −1.60 |
| −1.50 | .0559 | .0571 | .0582 | .0594 | .0606 | .0618 | .0630 | .0643 | .0655 | .0668 | −1.50 |
| −1.40 | .0681 | .0694 | .0708 | .0721 | .0735 | .0749 | .0764 | .0778 | .0793 | .0808 | −1.40 |
| −1.30 | .0823 | .0838 | .0853 | .0869 | .0885 | .0901 | .0918 | .0934 | .0951 | .0968 | −1.30 |
| −1.20 | .0985 | .1003 | .1020 | .1038 | .1056 | .1075 | .1093 | .1112 | .1131 | .1151 | −1.20 |
| −1.10 | .1170 | .1190 | .1210 | .1230 | .1251 | .1271 | .1292 | .1314 | .1335 | .1357 | −1.10 |
| −1.00 | .1379 | .1401 | .1423 | .1446 | .1469 | .1492 | .1515 | .1539 | .1562 | .1587 | −1.00 |
| −0.90 | .1611 | .1635 | .1660 | .1685 | .1711 | .1736 | .1762 | .1788 | .1814 | .1841 | −0.90 |
| −0.80 | .1867 | .1894 | .1922 | .1949 | .1977 | .2005 | .2033 | .2061 | .2090 | .2119 | −0.80 |
| −0.70 | .2148 | .2177 | .2206 | .2236 | .2266 | .2296 | .2327 | .2358 | .2389 | .2420 | −0.70 |
| −0.60 | .2451 | .2483 | .2514 | .2546 | .2578 | .2611 | .2643 | .2676 | .2709 | .2743 | −0.60 |
| −0.50 | .2776 | .2810 | .2843 | .2877 | .2912 | .2946 | .2981 | .3015 | .3050 | .3085 | −0.50 |
| −0.40 | .3121 | .3156 | .3192 | .3228 | .3264 | .3300 | .3336 | .3372 | .3409 | .3446 | −0.40 |
| −0.30 | .3483 | .3520 | .3557 | .3594 | .3632 | .3669 | .3707 | .3745 | .3783 | .3821 | −0.30 |
| −0.20 | .3859 | .3897 | .3936 | .3974 | .4013 | .4052 | .4090 | .4129 | .4168 | .4207 | −0.20 |
| −0.10 | .4247 | .4286 | .4325 | .4364 | .4404 | .4443 | .4483 | .4522 | .4562 | .4602 | −0.10 |
| 0.00 | .4641 | .4681 | .4721 | .4761 | .4801 | .4840 | .4880 | .4920 | .4960 | .5000 | 0.00 |

## Normal Distribution Table

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 | 0.00 |
| 0.10 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 | 0.10 |
| 0.20 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 | 0.20 |
| 0.30 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 | 0.30 |
| 0.40 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 | 0.40 |
| 0.50 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 | 0.50 |
| 0.60 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 | 0.60 |
| 0.70 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 | 0.70 |
| 0.80 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 | 0.80 |
| 0.90 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 | 0.90 |
| 1.00 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 | 1.00 |
| 1.10 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 | 1.10 |
| 1.20 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 | 1.20 |
| 1.30 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 | 1.30 |
| 1.40 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 | 1.40 |
| 1.50 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 | 1.50 |
| 1.60 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 | 1.60 |
| 1.70 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 | 1.70 |
| 1.80 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 | 1.80 |
| 1.90 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 | 1.90 |
| 2.00 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 | 2.00 |
| 2.10 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 | 2.10 |
| 2.20 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 | 2.20 |
| 2.30 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 | 2.30 |
| 2.40 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 | 2.40 |
| 2.50 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 | 2.50 |
| 2.60 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 | 2.60 |
| 2.70 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 | 2.70 |
| 2.80 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 | 2.80 |
| 2.90 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 | 2.90 |
| 3.00 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 | 3.00 |
| 3.10 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 | 3.10 |
| 3.20 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 | 3.20 |
| 3.30 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 | 3.30 |
| 3.40 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 | 3.40 |
| 3.50 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | 3.50 |
| 3.60 | .9998 | .9998 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | 3.60 |
| 3.70 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | 3.70 |
| 3.80 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | 3.80 |

**Q3.** (a) (i) Let $\{X_n\}_{n \geq 1}$ be a strictly decreasing sequence of positive random variables and suppose that $X_n \xrightarrow{p} 0$. Show that $X_n \xrightarrow{a.s.} 0$. *7*

(ii) Let X be a random variable defined with the probability function given by

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, 0 < x < \infty; 0 < \theta < \infty.$$

Given a random sample $(X_1, X_2, ..., X_n)$ of size n drawn from the population having the above probability density function, show that the sample mean $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ is the minimum variance bound estimator for $\theta$ and has variance $\frac{\theta^2}{n}$. *8*

(b) Let $X_1, X_2, ..., X_n$ be independent and identically distributed $U(0, \theta)$ random variables, where $\theta > 0$ is an unknown parameter. Show that

$$\max\{X_1, X_2, ..., X_n\} \xrightarrow{p} \theta.$$ *15*

(c) Define Monotone Likelihood Ratio (MLR) property. State its significance. Examine whether MLR property holds for the family of distributions when a random sample of n observations is drawn on X which follows a binomial distribution. *10*

**Q4.** (a) (i) Let $X_1, X_2, ...$ be a sequence of independent and identically distributed random variables with common mean $\mu$ and variance $\sigma^2$. Also, let $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \overline{X})^2$. Show that $\sqrt{n}\,(\overline{X} - \mu)/S \xrightarrow{L} Z$, where $Z \sim N(0, 1)$. *7*

(ii) Let p be the probability of getting a success in a Bernoulli trial. Assume that five independent Bernoulli trials are conducted for testing the hypothesis $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{3}{4}$. Find the size and power of the statistical test under the condition that $H_0$ is rejected if more than three successes occur. *4+4=8*

(b) (i) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A, B, C \in \Omega$ with P(B) and P(C) > 0. If B and C are independent, show that

$$P(A \mid B) = P(A \mid B \cap C)\, P(C) + P(A \mid B \cap C^c)\, P(C^c).$$ *7*

(ii) Let X and Y be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

(1) If $P(X \geq 0) = 1$, then show that $E(X \mid Y) \geq 0$ with probability 1.

(2) If $X_1$ and $X_2$ are two random variables defined on the same probability space and if $P(X_1 \geq X_2) = 1$, then show that $E(X_1 \mid Y) \geq E(X_2 \mid Y)$ with probability 1. *8*

(c) Define sufficiency of an estimator. Let $X_1$, $X_2$ and $X_3$ be three independent observations on a random variable X which follows a Bernoulli distribution with parameter θ. Examine the sufficiency of the statistic $T = X_1 + 2X_2 + 3X_3$ for θ. *10*

## SECTION B

**Q5.** (a) What do you mean by canonical correlations ? If $\mathbf{x} : r \times 1$ and $\mathbf{y} : s \times 1$ be two vectors such that $r \leq s$, Cov $(\mathbf{x}) = \Sigma_{11}$, Cov $(\mathbf{y}) = \Sigma_{22}$, Cov $(\mathbf{x}, \mathbf{y}) = \Sigma_{12}$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ has full rank. How many canonical correlations exist ? If $\lambda_i$ is an eigenvalue of $\Sigma_{11}^{-1/2} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1/2}$ with associated eigenvector $\mathbf{e}_i$, then show that $\lambda_i$ is also an eigenvalue of $\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$ with associated eigenvector $\Sigma_{11}^{-1/2} \mathbf{e}_i$. *8*

(b) Suppose that $\rho$ is a correlation matrix of order five with all off-diagonal elements equal to 0·80. Calculate the proportion of the total population variance explained by the first principal component. *8*

(c) From a list of 2400 minor household expenditures, a simple random sample of 180 items was drawn in order to estimate the total amount spent for operation of the household. Certain types of expenditures (on clothing and car upkeep) were not considered relevant. Of the 180 sample items, 150 were relevant. The sum and uncorrelated sum of squares of the relevant amounts (in ₹) were as follows :

$$\Sigma y_i = 433{\cdot}53, \; \Sigma y_i^2 = 1941{\cdot}83.$$

Estimate the total expenditure for household operation and find the standard error of the estimate. Note that $y_i$ represents household expenditure for $i^{th}$ relevant sample item. *8*

(d) What is meant by a uniformity trial ? How would you study the relative efficiency of RBD (Randomized Block Design) over CRD (Completely Randomized Design) using uniformity trial ? Obtain the expressions for estimated relative efficiency of RBD over CRD. *8*

(e) Find **b** and A so that the joint density function

$$f(x, y) = \frac{1}{2\cdot4\,\pi} \exp\left\{-\frac{1}{0\cdot72}\left(\frac{x^2}{4} - 0\cdot8xy + y^2\right)\right\}, -\infty < x, y < \infty$$

can be written in the form ($\mathbf{x}' = (x, y)$)

$$\frac{|A|^{1/2}}{(2\pi)^{p/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{b})' A (\mathbf{x}-\mathbf{b})\right\}.$$

Also find $\mu_x$, $\mu_y$, $\sigma_x$, $\sigma_y$ and $\rho_{xy}$. *8*

**Q6.** (a) Let $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$ be independent and identically distributed $3 \times 1$ random vectors with

$$\boldsymbol{\mu} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

If $\mathbf{Y}_1 = \frac{1}{2}\mathbf{X}_1 + \frac{1}{2}\mathbf{X}_2 + \frac{1}{2}\mathbf{X}_3 + \frac{1}{2}\mathbf{X}_4$, and

$$\mathbf{Y}_2 = \frac{1}{2\sqrt{3}}\mathbf{X}_1 + \frac{1}{2\sqrt{3}}\mathbf{X}_2 + \frac{1}{2\sqrt{3}}\mathbf{X}_3 - \frac{\sqrt{3}}{2}\mathbf{X}_4$$

be linear combination of vectors, then obtain (i) $E(\mathbf{Y}_1)$, (ii) $E(\mathbf{Y}_2)$, (iii) Var ($\mathbf{Y}_1$), (iv) Var ($\mathbf{Y}_2$), and (v) Cov ($\mathbf{Y}_1$, $\mathbf{Y}_2$). If $\mathbf{X}_i$ ( i = 1, 2, 3, 4) has trivariate normal distribution, then what is the joint distribution of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ ?

Further if $\mathbf{X} \sim \mathbf{N}_3(\boldsymbol{\mu}, \Sigma)$, then write the joint distribution of $(\mathbf{X}_1 - \mathbf{X}_2 + \mathbf{X}_3)$ and $(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{X}_3)$. Here $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{pmatrix}$, $\boldsymbol{\mu}$ and $\Sigma$ are as given above. *15*

(b) Nelson's goal was to estimate the lifetime (in hours) of an encapsulating resin for gold-aluminium bonds in integrated circuits operating at 120°C. Thirty-seven units were assigned at random to one of the five different temperature stresses, ranging from 175° to 250°. Following table gives the $\log_{10}$ (lifetimes) in hours for the test units.

| $\log_{10}$ times till failure of a resin under stress | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Temperature (°C) | | | | | | | | | |
| 175° | | 194° | | 213° | | 231° | | 250° | |
| 2·04 | 1·85 | 1·66 | 1·66 | 1·53 | 1·35 | 1·15 | 1·21 | 1·26 | 1·02 |
| 1·91 | 1·96 | 1·71 | 1·61 | 1·54 | 1·27 | 1·22 | 1·28 | 0·83 | 1·09 |
| 2·00 | 1·88 | 1·42 | 1·55 | 1·38 | 1·26 | 1·17 | 1·17 | 1·08 | 1·06 |
| 1·92 | 1·90 | 1·76 | 1·66 | 1·31 | 1·38 | 1·16 | | | |

Compute the overall mean and treatment effects. Compute the Analysis of Variance table. Write your conclusions.

(F-distribution table is provided at the end of the booklet). *15*

(c) Three independent measurements on each of the angles A, B and C of a triangle are as follows :

| A | B | C |
|---|---|---|
| 39·5 | 60·3 | 80·1 |
| 39·3 | 62·2 | 80·3 |
| 39·6 | 60·1 | 80·4 |

Obtain the best estimates of the three angles taking into account the relation that sum of angles is equal to 180°. *10*

**Q7.** (a) Explain Invariance property of Hotelling's $T^2$ statistic. Verify this property for testing $H_0 : \mu' = (9, 5)$ vs $H_1 : \mu' \neq (9, 5)$ using data matrix for a random sample of size n = 3 from a bivariate normal population

$$X = \begin{pmatrix} 6 & 9 \\ 10 & 6 \\ 8 & 3 \end{pmatrix}$$

and matrix $C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Test $H_0$ at $\alpha = 0{\cdot}05$ level of significance. Write your conclusion.

(Table of F-distribution percentage points ($\alpha = 0{\cdot}05$) is attached at the end of the booklet). *15*

(b) Explicitly state the main objective of randomized complete block design. State the corresponding additive model and the associated assumptions. Describe the procedure for testing the treatment effects when such a design is used. *15*

(c) How does curvilinear regression differ from linear regression ? Given

$$f(x, y) = \begin{cases} 1, & \text{if } |y| < x, \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

show that the regression of Y on X is linear but regression of X on Y is curvilinear. *10*

**Q8.** (a) Following table gives the data for 10 systematic samples with n = 4 and N = 40. Each column represents a systematic sample and the rows are the strata.

| | Systematic sample numbers | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Strata | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| I | 0 | 1 | 1 | 2 | 5 | 4 | 7 | 7 | 8 | 6 |
| II | 6 | 8 | 9 | 10 | 13 | 12 | 15 | 16 | 16 | 17 |
| III | 18 | 19 | 20 | 20 | 24 | 23 | 25 | 28 | 29 | 27 |
| IV | 26 | 30 | 31 | 31 | 33 | 32 | 35 | 37 | 38 | 38 |

Compare the precision of systematic sampling with simple random sampling. *15*

(b) To construct a procedure for detecting potential hemophilia A carriers, blood samples were assayed for two groups of women and measurements on two variables, $X_1 = \log_{10}$ (AHF activity) and $X_2 = \log_{10}$ (AHF-like antigen) recorded. After investigation, we have the following information. Here $n_1 = 30$, $n_2 = 22$.

$$\bar{\mathbf{x}}_1 = \begin{pmatrix} -0{\cdot}0065 \\ -0{\cdot}0390 \end{pmatrix}, \; \bar{\mathbf{x}}_2 = \begin{pmatrix} -0{\cdot}2483 \\ 0{\cdot}0262 \end{pmatrix},$$

$$S^{-1}_{\text{Pooled}} = \begin{pmatrix} 131{\cdot}158 & -90{\cdot}423 \\ -90{\cdot}423 & 108{\cdot}147 \end{pmatrix}.$$

Obtain Fisher's linear discriminant function. Classify the new observation $(-0{\cdot}210, -0{\cdot}044)$ in one of the two groups when $P_1 = P_2$ and $C(1|2) = C(2|1)$. If $C(1|2)$ = cost incurred when a $\pi_2$ observation is incorrectly classified as $\pi_1 = 10$ and $C(2|1)$ = cost incurred when a $\pi_1$ observation is incorrectly classified as $\pi_2 = 5$, then classify the new observation $(-0{\cdot}210, -0{\cdot}044)$ in one of the two groups $\pi_1$ or $\pi_2$. ($P_1$ = prior probability for group I, $P_2$ = prior probability for group II and $C(1|2)$ and $C(2|1)$ are cost of misclassification). *15*

(c) A population consists of five numbers : 2, 3, 6, 8, 11. Consider all possible samples of size 2 that can be drawn without replacement from this population. Find the following : *10*

(i) The mean of the population

(ii) The standard deviation of the population

(iii) The mean of the sampling distribution of means

(iv) The standard error of means

## F-distribution Table

$F_{0.95}$

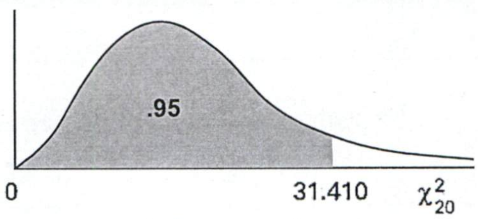
| Denominator Degrees of Freedom | Numerator Degrees of Freedom | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 236.8 | 238.9 | 240.5 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.17 | 2.09 | 2.02 | 1.96 |
| ∞ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 |

## F-distribution Table

$F_{0.99}$

| Denominator Degrees of Freedom | Numerator Degrees of Freedom | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
| 1 | 241.9 | 243.9 | 245.9 | 248.0 | 249.1 | 250.1 | 251.1 | 252.2 | 253.3 | 254.3 |
| 2 | 19.40 | 19.41 | 19.43 | 19.45 | 19.45 | 19.46 | 19.47 | 19.48 | 19.49 | 19.50 |
| 3 | 8.79 | 8.74 | 8.70 | 8.66 | 8.64 | 8.62 | 8.59 | 8.57 | 8.55 | 8.53 |
| 4 | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 | 5.66 | 5.63 |
| 5 | 4.74 | 4.68 | 4.62 | 4.56 | 4.53 | 4.50 | 4.46 | 4.43 | 4.40 | 4.36 |
| 6 | 4.06 | 4.00 | 3.94 | 3.87 | 3.84 | 3.81 | 3.77 | 3.74 | 3.70 | 3.67 |
| 7 | 3.64 | 3.57 | 3.51 | 3.44 | 3.41 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
| 8 | 3.35 | 3.28 | 3.22 | 3.15 | 3.12 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
| 9 | 3.14 | 3.07 | 3.01 | 2.94 | 2.90 | 2.86 | 2.83 | 2.79 | 2.75 | 2.71 |
| 10 | 2.98 | 2.91 | 2.85 | 2.77 | 2.74 | 2.70 | 2.66 | 2.62 | 2.58 | 2.54 |
| 11 | 2.85 | 2.79 | 2.72 | 2.65 | 2.61 | 2.57 | 2.53 | 2.49 | 2.45 | 2.40 |
| 12 | 2.75 | 2.69 | 2.62 | 2.54 | 2.51 | 2.47 | 2.43 | 2.38 | 2.34 | 2.30 |
| 13 | 2.67 | 2.60 | 2.53 | 2.46 | 2.42 | 2.38 | 2.34 | 2.30 | 2.25 | 2.21 |
| 14 | 2.60 | 2.53 | 2.46 | 2.39 | 2.35 | 2.31 | 2.27 | 2.22 | 2.18 | 2.13 |
| 15 | 2.54 | 2.48 | 2.40 | 2.33 | 2.29 | 2.25 | 2.20 | 2.16 | 2.11 | 2.07 |
| 16 | 2.49 | 2.42 | 2.35 | 2.28 | 2.24 | 2.19 | 2.15 | 2.11 | 2.06 | 2.01 |
| 17 | 2.45 | 2.38 | 2.31 | 2.23 | 2.19 | 2.15 | 2.10 | 2.06 | 2.01 | 1.96 |
| 18 | 2.41 | 2.34 | 2.27 | 2.19 | 2.15 | 2.11 | 2.06 | 2.02 | 1.97 | 1.92 |
| 19 | 2.38 | 2.31 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 | 1.93 | 1.88 |
| 20 | 2.35 | 2.28 | 2.20 | 2.12 | 2.08 | 2.04 | 1.99 | 1.95 | 1.90 | 1.84 |
| 21 | 2.32 | 2.25 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 | 1.87 | 1.81 |
| 22 | 2.30 | 2.23 | 2.15 | 2.07 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.78 |
| 23 | 2.27 | 2.20 | 2.13 | 2.05 | 2.01 | 1.96 | 1.91 | 1.86 | 1.81 | 1.76 |
| 24 | 2.25 | 2.18 | 2.11 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.79 | 1.73 |
| 25 | 2.24 | 2.16 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 | 1.77 | 1.71 |
| 26 | 2.22 | 2.15 | 2.07 | 1.99 | 1.95 | 1.90 | 1.85 | 1.80 | 1.75 | 1.69 |
| 27 | 2.20 | 2.13 | 2.06 | 1.97 | 1.93 | 1.88 | 1.84 | 1.79 | 1.73 | 1.67 |
| 28 | 2.19 | 2.12 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 | 1.71 | 1.65 |
| 29 | 2.18 | 2.10 | 2.03 | 1.94 | 1.90 | 1.85 | 1.81 | 1.75 | 1.70 | 1.64 |
| 30 | 2.16 | 2.09 | 2.01 | 1.93 | 1.89 | 1.84 | 1.79 | 1.74 | 1.68 | 1.62 |
| 40 | 2.08 | 2.00 | 1.92 | 1.84 | 1.79 | 1.74 | 1.69 | 1.64 | 1.58 | 1.51 |
| 60 | 1.99 | 1.92 | 1.84 | 1.75 | 1.70 | 1.65 | 1.59 | 1.53 | 1.47 | 1.39 |
| 120 | 1.91 | 1.83 | 1.75 | 1.66 | 1.61 | 1.55 | 1.50 | 1.43 | 1.35 | 1.25 |
| ∞ | 1.83 | 1.75 | 1.67 | 1.57 | 1.52 | 1.46 | 1.39 | 1.32 | 1.22 | 1.00 |

## Percentiles of the Chi-Square Distribution




$p(\chi^2_{20} \leq 31.410) = .95$

| d.f. | $\chi^2_{.005}$ | $\chi^2_{.025}$ | $\chi^2_{.05}$ | $\chi^2_{.90}$ | $\chi^2_{.95}$ | $\chi^2_{.975}$ | $\chi^2_{.99}$ | $\chi^2_{.995}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | .0000393 | .000982 | .00393 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 |
| 2 | .0100 | .0506 | .103 | 4.605 | 5.991 | 7.378 | 9.210 | 10.597 |
| 3 | .0717 | .216 | .352 | 6.251 | 7.815 | 9.348 | 11.345 | 12.838 |
| 4 | .207 | .484 | .711 | 7.779 | 9.488 | 11.143 | 13.277 | 14.860 |
| 5 | .412 | .831 | 1.145 | 9.236 | 11.070 | 12.832 | 15.086 | 16.750 |
| 6 | .676 | 1.237 | 1.635 | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7 | .989 | 1.690 | 2.167 | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8 | 1.344 | 2.180 | 2.733 | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9 | 1.735 | 2.700 | 3.325 | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10 | 2.156 | 3.247 | 3.940 | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11 | 2.603 | 3.816 | 4.575 | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12 | 3.074 | 4.404 | 5.226 | 18.549 | 21.026 | 23.336 | 26.217 | 28.300 |
| 13 | 3.565 | 5.009 | 5.892 | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| 14 | 4.075 | 5.629 | 6.571 | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| 15 | 4.601 | 6.262 | 7.261 | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| 16 | 5.142 | 6.908 | 7.962 | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| 17 | 5.697 | 7.564 | 8.672 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| 18 | 6.265 | 8.231 | 9.390 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| 19 | 6.844 | 8.907 | 10.117 | 27.204 | 30.144 | 32.852 | 36.191 | 38.582 |
| 20 | 7.434 | 9.591 | 10.851 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| 21 | 8.034 | 10.283 | 11.591 | 29.615 | 32.671 | 35.479 | 38.932 | 41.401 |
| 22 | 8.643 | 10.982 | 12.338 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| 23 | 9.260 | 11.688 | 13.091 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| 24 | 9.886 | 12.401 | 13.848 | 33.196 | 36.415 | 39.364 | 42.980 | 45.558 |
| 25 | 10.520 | 13.120 | 14.611 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| 26 | 11.160 | 13.844 | 15.379 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| 27 | 11.808 | 14.573 | 16.151 | 36.741 | 40.113 | 43.194 | 46.963 | 49.645 |
| 28 | 12.461 | 15.308 | 16.928 | 37.916 | 41.337 | 44.461 | 48.278 | 50.993 |
| 29 | 13.121 | 16.047 | 17.708 | 39.087 | 42.557 | 45.722 | 49.588 | 52.336 |
| 30 | 13.787 | 16.791 | 18.493 | 40.256 | 43.773 | 46.979 | 50.892 | 53.672 |
| 35 | 17.192 | 20.569 | 22.465 | 46.059 | 49.802 | 53.203 | 57.342 | 60.275 |
| 40 | 20.707 | 24.433 | 26.509 | 51.805 | 55.758 | 59.342 | 63.691 | 66.766 |
| 45 | 24.311 | 28.366 | 30.612 | 57.505 | 61.656 | 65.410 | 69.957 | 73.166 |
| 50 | 27.991 | 32.357 | 34.764 | 63.167 | 67.505 | 71.420 | 76.154 | 79.490 |
| 60 | 35.535 | 40.482 | 43.188 | 74.397 | 79.082 | 83.298 | 88.379 | 91.952 |
| 70 | 43.275 | 48.758 | 51.739 | 85.527 | 90.531 | 95.023 | 100.425 | 104.215 |
| 80 | 51.172 | 57.153 | 60.391 | 96.578 | 101.879 | 106.629 | 112.329 | 116.321 |
| 90 | 59.196 | 65.647 | 69.126 | 107.565 | 113.145 | 118.136 | 124.116 | 128.299 |
| 100 | 67.328 | 74.222 | 77.929 | 118.498 | 124.342 | 129.561 | 135.807 | 140.169 |